B.Sc Part II (Hons)

Homomorphism of Rings

Definition:– Let $(R, +, \cdot)$ and $(R', \oplus, \odot)$ be two rings. Let $f: R \to R'$ be a mapping such that

$$f(a+b) = f(a) \oplus f(b)$$

$$f(a \cdot b) = f(a) \odot f(b)$$

Then f is said to be a homomorphism of the ring R into the ring R. If it is a one-one onto mapping, then f is said to be an isomorphism and R' and R are said to be isomorphic rings.

Example– Consider the set of integers I and the set of even integers E. We are going to prove that the integers and even integers are not isomorphic as rings.

Let $f: I \to E$ be defined by $f(n) = 2n$

Let $m, n \in I$. Then

$$f(m+n) = 2(m+n) = 2m + 2n$$
$$= f(m) + f(n)$$

and $f(m \cdot n) = 2mn$ which is $\neq f(m) \cdot f(n)$

Hence f is not a homomorphism and consequently not an isomorphism.

Theorem (A):– Let $f: R \to R'$ be a homomorphism of a ring R onto ring R'.

If 1 is the unit element of R and $f(1) = 1'$ prove that $1'$ is the unit element of R'.

Proof:– Since f is a homomorphism of R onto R', therefore R' is a homomorphic image of R. If 1 is the unity element of R, then $f(1) \in R'$.

Let a' be any element of R'.

Then $f(a) = a'$ for some $a \in R$ since f is onto R'.

We have $f(1)a' = f(1)f(a)$ since $a' = f(a)$

$$= f(1a) = f(a) = a'$$

and $a'f(1) = f(a)f(1) = f(a1) = f(a) = a'$

$\therefore f(1)$ is the unity element of R'.

Thus if $f(1) = 1'$, then $1'$ is the unity element of R'.

Theorem (B):- If f be a homomorphism of a ring R into R^* and if R and R^* possesses unity elements 1 and 1^* respectively, then $f(1) = 1^*$ if

(i) f is R onto R^* or (ii) If R^* is an integral domain.

Solution _ (i) This is already proved in the above theorem.

(ii) Let f is a homomorphism of a ring R into an integral domain R^*.

Let there be an element $a \in R$ such that $f(a) = 0 \in R^*$.

We have $f(1) f(a) = f(1a) = a$.

Now let b' be any element of R^*

We have, $f(a) b' = f(a) b'$

$\Rightarrow f(1) f(a) b' = f(a) b' \quad \because f(1) f(a) = f(a)$

$\Rightarrow f(a) [f(1) b'] = f(a) b'$

$\because f(1), f(a) \in R^*$ Which being an integral domain, is a commutative ring.

$\Rightarrow f(a) [f(1) b'] - f(a) b' = 0$

$\Rightarrow f(a) [f(1) b' - b'] = 0$

$\Rightarrow f(a) [f(1) b' - b'] = 0$

$\Rightarrow f(1) b' - b' = 0 \quad \because f(a) \neq 0$ and R^* is without zero divisors.

$\Rightarrow f(1) b' = b' = b' f(1) \quad \because R^*$ is a Commutative ring.

Thus $f(1) b' = b' = b' f(1)$ for all $b' \in R^*$

$\therefore f(1)$ is the unit element of R^*.

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